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XIII.

ON REAL ORTHOGONAL SUBSTITUTION.

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§ 1. Real Proper Orthogonal Matrices.

1. In a paper to appear in a forthcoming number of the "Quarterly Journal of Mathematics" I have shown that, if ϕ is any real proper orthogonal matrix, then, for a proper choice of the real skew symmetric matrix θ, we may put $\phi = e^{\theta}$, where e^{θ} denotes the exponential series $\Sigma_r \frac{\theta^r}{r!}$, which is convergent for any matrix. This theorem was published in these Proceedings, Vol. XXVI. It follows immediately from this theorem that any real orthogonal matrix whatever is given by the expression

$$\omega \left(\frac{1 - \Upsilon}{1 + \Upsilon}\right)^2,$$

for a proper choice of the real skew symmetric matrix Υ and of the matrix ω whose constituents are all zero except those in the principal diagonal which are severally equal to ± 1. The second factor in the above expression is the square of Cayley's expression for an orthogonal matrix.

If the determinant of the orthogonal matrix is equal to unity, we may put $\omega = 1$; if the determinant is equal to -1, and if at the same time unity is a latent root of even multiplicity,[*] we may put $\omega = -1$.

I shall show in this section that, as a consequence of the exponential representation of real proper orthogonal matrices, any such matrix may be represented by the square of Cayley's expression; and in § 2 it will be shown that the general theorem given above follows as a consequence of the theorem just stated.

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[*] If unity is not a latent root of the orthogonal matrix, its multiplicity is zero.

2. Let ϕ be a real proper orthogonal matrix, then by the theorem above referred to we may put

$$\phi = e^{\theta},$$

where θ is a real skew symmetric matrix.

Since θ is skew symmetric, its latent roots occur in pairs opposite in sign; that is, if H is a latent root of θ, then $-H$ is also a latent root of θ having the same multiplicity as H; since θ is skew symmetric and real, its latent roots are purely imaginary.*

It may be that among the latent roots of θ are integer multiples of $2\pi\sqrt{-1}$; in this case a real skew symmetric matrix θ_1 can always be found of which no integer multiple of $2\pi\sqrt{-1}$ is a latent root, and such that

$$\phi = e^{\theta} = e^{\theta_1}.$$

Thus, let the latent roots of θ be given by the following schedule:

Latent root,	$h_0 = 0$	$\pm h_1\sqrt{-1}$	$\pm h_2\sqrt{-1}$	....
Multiplicity,	m_0	m_1	m_2	....

Latent root,	$\pm h_\mu\sqrt{-1}$	$\pm h_{\mu+1}\sqrt{-1}$	....	$\pm h_\nu\sqrt{-1}$,
Multiplicity,	m_μ	$m_{\mu+1}$	....	m_ν,

in which m_0 denotes the multiplicity of the latent root $h_0 = 0$, m_1 denotes the multiplicity of each of the latent roots $\pm h_1\sqrt{-1}$, etc. Let $h_1, h_2, \ldots h_\mu$, be integer multiples of 2π, and $h_{\mu+1}, \ldots h_\nu$, any real quantities other than integer multiples of 2π. Since θ is real, its identical equation is then, if $m_0 \neq 0$,

$$F(\theta) \equiv \theta(\theta^2 + h_1^2)(\theta^2 + h_2^2)\ldots\ldots(\theta^2 + h_\nu^2) = 0.\dagger$$

Let x be any scalar, and let $f^{(1)}_{(r)}x$ and $f^{(2)}_r(x)$ be defined as follows:

$$f^{(1)}_r(x) = \left(\frac{F(x)}{x - h_r\sqrt{-1}}\right) : \left(\frac{F(x)}{x - h_r\sqrt{-1}}\right)_{x = h_r\sqrt{-1}},$$

$$f^{(2)}_r(x) = \left(\frac{F(x)}{x + h_r\sqrt{-1}}\right) : \left(\frac{F(x)}{x + h_r\sqrt{-1}}\right)_{x = -h_r\sqrt{-1}},$$

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* Proc. Lond. Math. Soc., Vol. XXII. p. 453.

† Ibid., p. 462.

for $r = 0, 1, 2, \ldots . \mu, \ldots . \nu$. Since $f_0^{(1)}(x) = f_0^{(2)}(x)$, either of these two functions may be denoted simply by $f_0(x)$. We then have

$$f_0(\theta) + f_1^{(1)}(\theta) + f_1^{(2)}(\theta) + \ldots . f_\nu^{(1)}(\theta) + f_\nu^{(2)}(\theta) = 1;$$

$$(f_r^{(p)}(\theta))^2 = f_r^{(p)}(\theta),$$

for $p = 1, 2$, and $r = 0, 1, 2, \ldots . \mu, \ldots . \nu$;

$$f_r^{(p)}(\theta) \cdot f_s^{(q)}(\theta) = 0,$$

for $p, q = 1, 2$, and $r, s = 0, 1, 2, \ldots . \mu, \ldots . \nu$, but $r \ddagger s$; and

$$\begin{aligned} \text{tr.} f_0(\theta) &= f_0(\text{tr.}\, \theta) \\ &= f_0(-\theta) \\ &= f_0(\theta), \\ \text{tr.} f_r^{(1)}(\theta) &= f_r^{(1)}(\text{tr.}\, \theta) \\ &= f_r^{(1)}(-\theta) \\ &= f_r^{(2)}(\theta), \end{aligned}$$

for $r = 1, 2, 3 \ldots . \mu, \ldots . \nu$. We also have

$$\theta = 0 \,. f_0(\theta) + h_1 \sqrt{-1} f_1^{(1)} \theta - h_1 \sqrt{-1} f_1^{(2)} \theta + \ldots . \\ + h_\nu \sqrt{-1} f_\nu^{(1)}(\theta) - h_\nu \sqrt{-1} f_\nu^{(2)}(\theta);$$

therefore, if $f(\theta)$ denotes any polynomial in powers of θ or convergent power series in θ,

$$f(\theta) = f 0 \,. f_0(\theta) + f(h_1 \sqrt{-1}) f_1^{(1)}(\theta) + f(-h_1 \sqrt{-1}) f_1^{(2)}(\theta) \\ + \ldots . + f(h_\nu \sqrt{-1}) f_\nu^{(1)}(\theta) + f(-h_\nu \sqrt{-1}) f_\nu^{(2)}(\theta).$$

Thus

$$\phi = e^\theta = e^0 f_0(\theta) + e^{h_1\sqrt{-1}} f_1^{(1)}(\theta) + e^{-h_1\sqrt{-1}} f_1^{(2)}(\theta) + \ldots . \\ + e^{h_\nu\sqrt{-1}} f_\nu^{(1)}(\theta) + e^{-h_\nu\sqrt{-1}} f_\nu^{(2)}(\theta).$$

From the relations given above between the functions with the same subscript it is evident that this matrix is orthogonal.

Let now

$$\begin{aligned}\theta_1 = \theta - \big[2\, h_1 \sqrt{-1}\, &\big(f_1^{(1)}(\theta) - f_1^{(2)}(\theta)\big) \\ &+ 2\, h_2 \sqrt{-1}\, \big(f_2^{(1)}(\theta) - f_2^{(2)}(\theta)\big) + \ldots\ldots \\ &+ 2\, h_\mu \sqrt{-1}\, \big(f_\mu^{(1)}(\theta) - f_\mu^{(2)}(\theta)\big)\big] \\ = 0\,\big(f_0(\theta) + f_1^{(1)}(\theta) &+ f_1^{(2)}(\theta) + \ldots\ldots + f_\mu^{(1)}(\theta) + f_\mu^{(2)}(\theta)\big) \\ &+ h_{\mu+1}\sqrt{-1}\, f_{\mu+1}^{(1)}(\theta) - h_{\mu+1}\sqrt{-1}\, f_{\mu+1}^{(2)}(\theta) + \ldots\ldots \\ &+ h_\nu \sqrt{-1}\, f_\nu^{(1)}(\theta) - h_\nu \sqrt{-1}\, f_\nu^{(2)}(\theta).\end{aligned}$$

The matrix θ_1 is then skew symmetric, and its latent roots are 0, $\pm h_{\mu+1}\sqrt{-1}$, etc.; θ_1 is moreover real, since $f_{\mu+1}^{(1)}\theta - f_{\mu+1}^{(2)}(\theta)$, $f_{\mu+2}^{(1)}(\theta) - f_{\mu+2}^{(2)}(\theta)$, etc., are purely imaginary scalar multiples of real matrices. We also have

$$\phi = e^{\theta} = e^{\theta_1}.$$

If $m_0 = 0$, that is, if zero is not a latent root of θ, we may proceed in a precisely similar way to find the matrix θ_1.

3. Since no integer multiple of $2\pi\sqrt{-1}$, other than zero, is a latent root of θ_1, hence it follows that no odd multiple of $\pi\sqrt{-1}$ is a latent root of $\frac{\theta_1}{2}$; therefore, the matrix

$$\psi = e^{\frac{\theta_1}{2}}$$

has no latent root equal to -1. For the latent roots of ψ are contained in the expression e^H, for H equal successively to the distinct latent roots of θ_1.

We have

$$\psi^2 = \left(e^{\frac{\theta}{2}}\right)^2 = e^{\theta} = \phi.$$

Since θ_1 is real, ψ is real; and ψ is orthogonal, for we have

$$\begin{aligned}\psi\,.\,\mathrm{tr.}\,\psi &= e^{\frac{\theta_1}{2}}\,.\,e^{\mathrm{tr.}\frac{\theta_1}{2}} \\ &= e^{\frac{\theta_1}{2}}\,.\,e^{-\frac{\theta_1}{2}} \\ &= e^{\frac{\theta_1}{2}-\frac{\theta_1}{2}} \\ &= 1.\end{aligned}$$

Since —1 is not a latent root of ψ,

$$\left| \psi \right| = 1.$$

Or, since $e^{\frac{\theta_1}{4}}$ is orthogonal, and $\left(e^{\frac{\theta_1}{4}}\right)^2 = e^{\frac{\theta_1}{2}}$, therefore,

$$\left| \psi \right| = \left| e^{\frac{\theta_1}{2}} \right| = \left| e^{\frac{\theta_1}{4}} \right|^2 = 1.$$

Therefore, every real proper orthogonal matrix has among its square roots one or more real proper orthogonal matrices of which —1 is not a latent root.

4. Since

$$\left| \psi + 1 \right| \neq 0,$$

we may put

$$\Upsilon = \frac{1 - \psi}{1 + \psi};$$

in which case Υ will be real, and we shall have

$$\text{tr.}\,\Upsilon = \frac{1 - \text{tr.}\,\psi}{1 + \text{tr.}\,\psi} = \frac{1 - \psi^{-1}}{1 + \psi^{-1}} = \frac{\psi - 1}{\psi + 1} = -\Upsilon,$$

and also

$$\psi = \frac{1 - \Upsilon}{1 + \Upsilon}.$$

Therefore, we may put

$$\phi = \psi^2 = \left(\frac{1 - \Upsilon}{1 + \Upsilon}\right)^2,$$

for a proper choice of the skew symmetric matrix Υ.

§ 2. REAL IMPROPER ORTHOGONAL MATRICES.

5. If Φ is a real improper orthogonal matrix of which unity is a latent root of even multiplicity,* it is the negative of a real proper orthogonal matrix; therefore by (4) we may put

$$\Phi = -\left(\frac{1 - \Upsilon}{1 + \Upsilon}\right)^2,$$

for a proper choice of the real skew symmetric matrix Υ.

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* This includes the case in which unity is not a latent root of Φ; the multiplicity of unity is then zero.

6. Let Φ be any real improper orthogonal matrix. Then if ω denotes a matrix whose determinant is equal to -1, and whose constituents are all zero except those in the principal diagonal which are severally equal to ± 1, the matrix

$$\phi = \omega \Phi$$

is a real proper orthogonal matrix. For

$$\begin{aligned} \check{\phi}\phi &= \check{\Phi}\check{\omega} \,.\, \omega\Phi \\ &= \check{\Phi}\omega^2\Phi \\ &= \check{\Phi}\Phi \\ &= 1. \end{aligned}$$

Moreover,

$$| \phi | = | \omega | \,.\, | \Phi | = 1.$$

Therefore, we may put

$$\phi = \left(\frac{1 - \Upsilon}{1 + \Upsilon}\right)^2$$

for a proper choice of the real skew symmetric matrix Υ. Whence, since $\omega^2 = 1$, we derive

$$\Phi = \omega\phi = \omega\left(\frac{1 - \Upsilon}{1 + \Upsilon}\right)^2.$$

NOTE ON IMAGINARY ORTHOGONAL MATRICES.

Let ϕ be an imaginary proper orthogonal matrix whose distinct latent roots are $1, -1, g, g^{-1}$; and let the rational integral function of ϕ of lowest order that vanishes be

$$(\phi-1)^m (\phi+1) (\phi-g)^p (\phi-g^{-1})^p.$$

Let

$$A = \frac{[(\phi-1)^m-(-1-1)^m]\,[(\phi-1)^m-(g-1)^m]^p\,[(\phi-1)^m-(g^{-1}-1)^m]^p}{[-(-1-1)^m]\,[-(g-1)^m]^p\,[-(g^{-1}-1)^m]^p},$$

$$B = \frac{[(\phi+1)-(1+1)]^m\,[(\phi+1)-(g+1)]^p\,[(\phi+1)-(g^{-1}+1)]^p}{[-(1+1)]^m\,[-g+1)]^p\,[-(g^{-1}+1)]^p},$$

$$C = \frac{[(\phi-g)^p-(1-g)^p]^m\,[(\phi-g)^p-(-1-g)^p]\,[(\phi-g)^p-(g^{-1}-g)^p]^p}{[-(1-g)^p]^m\,[-(-1-g)^p]\,[-(g^{-1}-g)^p]^p};$$

let D be obtained from C by interchanging g and g^{-1} in the expression for C. Then

$$A + B + C + D = 1;$$

$$A^2 = A,\quad B^2 = B,\quad C^2 = C,\quad D^2 = D,$$

$$AB = BA = AC = CA = \dots. = 0,$$

(that is, all binary products formed from two different letters vanish);

$$\text{tr. } A = A,\quad \text{tr. } B = B,\quad \text{tr. } C = D,\quad \text{tr. } D = C.$$

Moreover,

$$\begin{aligned} (\phi-1)^{m-1} A &\neq 0, & (\phi-1)^m A &= 0, \\ & & (\phi+1)\, B &= 0, \\ (\phi-g)^{p-1}\, C &\neq 0, & (\phi-g)^p\, C &= 0, \\ (\phi-g^{-1})^{p-1} D &\neq 0, & (\phi-g^{-1})^p\, D &= 0. \end{aligned}$$

I find that the matrix B is separable into a sum of two matrices, B_1 and B_2, such that

$$B_1^2 = B_1, \qquad B_2^2 = B_2,$$

$$\text{tr. } B_2 = B_1,$$

$$B_1 B_2 = B_2 B_1 = 0.^*$$

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* The products of B_1 and B_2 by or into either of the letters A, C, or D also vanish. Thus,

$$(B_1 + B_2)\, A = BA = 0;$$

$$\therefore\ B_1 A = B_1 (B_1 + B_2)\, A = 0.$$

If now

$$\begin{aligned}\psi = {} & [1 + c_1 \overline{\phi - 1} + c_2 \overline{\phi - 1}^2 + \dots + c_{m-1} (\phi - 1)^{m-1}] A \\ & + \sqrt{-1}\, B_1 - \sqrt{-1}\, B_2 + \sqrt{g} \left[1 + c_1 \left(\frac{\phi - g}{g}\right) + c_2 \left(\frac{\phi - 1}{g}\right)^2 \right. \\ & \left. + \dots + c_{p-1} \left(\frac{\phi - 1}{g}\right)^{p-1}\right] C + (\sqrt{g})^{-1} \left[1 + c_1 \left(\frac{\phi - g^{-1}}{g^{-1}}\right) \right. \\ & \left. + c_2 \left(\frac{\phi - g^{-1}}{g^{-1}}\right)^2 + \dots + c^{p-1} \left(\frac{\phi - g^{-1}}{g^{-1}}\right)^{p-1}\right] D,\end{aligned}$$

where 1, c_1, c_2, etc. denote the coefficients of x in the expansion by the binomial theorem of $(1 + x)^{\frac{1}{2}}$; then -1 will not be a latent root of ψ, and we shall have

$$\psi^2 = \phi,$$

$$\psi \,.\, \text{tr.}\, \psi = 1.$$

Therefore, proceeding as in (4), it may be shown that we have

$$\phi = \left(\frac{1 - \Upsilon}{1 + \Upsilon}\right)^2$$

for a proper choice of the skew symmetric matrix Υ.

This proof may be extended to any imaginary proper orthogonal matrix for which the nullity of $\phi + 1$ is equal to the multiplicity of the latent root -1.

For any matrix ϕ whose determinant does not vanish (that is, of which zero is not a latent root), a matrix ϑ can always be found such that

$$\phi = e^{\vartheta}.$$

Let ϑ be determined by Sylvester's formula as a finite polynomial in powers of ϕ; thus let

$$\vartheta = f(\phi).$$

We then have

$$\text{tr.}\, \vartheta = f(\text{tr.}\, \phi);$$

whence, if ϕ is orthogonal, it follows that

$$\vartheta \,.\, \text{tr.}\, \vartheta = \text{tr.}\, \vartheta \,.\, \vartheta.$$

Let

$$\frac{\vartheta + \text{tr.}\, \vartheta}{2} = \theta_0, \qquad \frac{\vartheta - \text{tr.}\, \vartheta}{2} = \theta;$$

from the preceding equation it follows that

$$\theta_0 \,.\, \theta = \theta \,.\, \theta_0 \,;$$

therefore

$$\phi = e^{\vartheta} = e^{\theta_0 + \theta} = e^{\theta_0} e^{\theta}.$$

Since θ_0 is symmetric, e^{θ_0} is symmetric. We have

$$(e^{\theta_0})^2 = e^{2\theta_0} = e^{\vartheta + \mathrm{tr.}\,\vartheta} = e^{\vartheta} \,.\, e^{\mathrm{tr.}\,\vartheta} = \phi \,.\, \mathrm{tr.}\,\phi = 1\,;$$

therefore, the first factor in the above expression for ϕ is a symmetric square root of unity, that is, is a symmetric orthogonal matrix. Since θ is skew symmetric, e^{θ} is a proper orthogonal matrix. Moreover, e^{θ} can be represented by the square of Cayley's expression. For, if no integer multiple of $2\,\pi\,\sqrt{-1}$ is a latent root of θ, $e^{\frac{\theta}{2}}$ can be represented by Cayley's expression; if, on the contrary, there are integer multiples of $2\,\pi\,\sqrt{-1}$ among the latent roots of θ, a skew symmetric matrix θ_1, can always be found of which no integer multiple of $2\,\pi\,\sqrt{-1}$ is a latent root, and such that

$$e^{\theta} = e^{\theta_1}.$$

Therefore, in either case the theorem is true.

May 1, 1893.

NOTE ON SYMMETRIC ORTHOGONAL MATRICES.

Every symmetric orthogonal matrix is a symmetric square root of unity, and therefore, if Φ is a symmetric orthogonal matrix, an orthogonal matrix ϖ can always be found to satisfy the equation

$$\Phi = \varpi\,\omega\,\varpi^{-1},$$

in which ω is a matrix whose constituents are all zero except those in the principal diagonal which are severally equal to ± 1.

If Φ is real, ϖ may be taken real, and hence it follows from the theorems of § 1 and § 2 that, for a proper choice of the real skew symmetric matrix Υ, and of ω' a matrix similar to ω, we may put

$$\varpi = \left(\frac{1-\Upsilon}{1+\Upsilon}\right)^2 \omega'.$$

Therefore,

$$\Phi = \left(\frac{1-\Upsilon}{1+\Upsilon}\right)^2 \omega' \,.\, \omega \,.\, \omega' \left(\frac{1+\Upsilon}{1-\Upsilon}\right)^2 = \left(\frac{1-\Upsilon}{1+\Upsilon}\right)^2 \omega \left(\frac{1+\Upsilon}{1-\Upsilon}\right)^2.$$

August 17, 1893.